§2.2/2.3 Inverses and Invertible Matrices

Recall the identity matrix $I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$,

the $n \times n$ matrix with 1's on the diagonal and 0's elsewhere.

Defn: An $n \times n$ A matrix is invertible if there exists an $n \times n$ matrix B such that $AB = BA = I_n$. In this ~~ca~~ case B is called the inverse of A and we would denote this by $B = A^{-1}$.

Remark: Invertibility only occurs for square matrices so that both AB and BA are defined.

- If A has an inverse, A^{-1} it is unique.

Theorem 1: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix.
If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

we'll see this term again in §3.1

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$1 \cdot 4 - 2 \cdot 3 \neq 0$ so A is invertible

also

$$A^{-1} = \frac{1}{4-6}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Let's check:

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} (-2+3) & (1-1) \\ (-6+6) & (3-2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (-2+3) & (-4+4) \\ (\frac{3}{2}-\frac{3}{2}) & (3-2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Why do we care? Matrix inverses allow us to solve systems quickly.

Theorem 2

If A is an invertible $n \times n$ matrix, then for all b in $\mathbb{R}^n$, $Ax = b$ has a unique solution, which is $x = A^{-1}b$.

Proof

$$Ax = b \implies A^{-1}(Ax) = A^{-1}b \implies x = A^{-1}b$$

Remark

If A is $n \times n$ and invertible, consider $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$
where $T(x) = Ax$

1) Since $Ax = b$ has a solution for all b, T is onto

2) Since the solution is unique, T is one-to-one.
   - Can also see this as

$$T(x) = Ax = 0 \quad \Rightarrow \quad x = A^{-1} 0 = 0$$

<u>Defn</u>: If A is any $m \times n$ matrix, the transpose of A, A^T is the $n \times m$ matrix whose rows are the columns of A and whose columns are the rows of A. In other words

$$(A^T)_{ij} = (A)_{ji}$$

<u>Remark</u>

- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$

<u>Example</u>

- $$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

- $$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

- $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In general

$$(I_n)^T = I_n$$

Theorem 3

1) If A is invertible, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

2) If A and B are $n \times n$ invertible matrices, AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

3) If A is invertible, then A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

Proof

1) $A^{-1}A = AA^{-1} = I$ so A^{-1} has an inverse and $(A^{-1})^{-1} = A$

2) $(AB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$

$(B^{-1}A^{-1})(AB) = B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$

3) $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$

$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$

Question: How do we find inverses of $n \times n$ matrices when $n > 2$?

Defn: An elementary matrix is an $n \times n$ matrix which is obtained by performing a single elementary row operation on I_n

Examples

1) $E = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ $\qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to 3R_1} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

2) $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

3) $E = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ $\qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

who cares? Notice that if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

1) $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 3 & 4 \end{bmatrix}$ $\qquad A \xrightarrow{3 \cdot R_1} \begin{bmatrix} 3 & 6 \\ 3 & 4 \end{bmatrix}$

2) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ $\qquad A \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

3) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 5 & 8 \end{bmatrix}$ $\qquad A \xrightarrow{2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 2 \\ 5 & 8 \end{bmatrix}$

So if E corresponds to a particular elemen~~tary~~ row operation, then multiplying A on the left by E performs that operation on A.

Theorem 4

An $n \times n$ matrix A is invertible if and only if it is row equivalent to I_n.

As it turns out, any sequence of row operations that row reduces A to I_n also ~~more~~ transforms I_n to A^{-1}

$$A \xrightarrow{\text{some row operations}} I_n \qquad I_n \xrightarrow{\text{the same row operations}} A^{-1}$$

In other words

$$\text{if} \quad \underbrace{E_p \cdots\cdots E_1}_{= A^{-1}} A = I_n$$

$$\Rightarrow \quad = A^{-1}$$

So all we have to do is row reduce A to I_n and keep track of the row operations. ~~In~~ In other words, if

$$[\, A \mid I_n \,] \overset{\text{row equivalent}}{\sim} [\, I_n \mid B \,]$$

Then $B = A^{-1}$

Example

Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -2 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -24 & 18 & 5 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & -4 & 1 \end{array}\right]$$

Thus $A^{-1} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & -4 & 1 \end{bmatrix}$

Notice that since $I = [e_1 | \cdots | e_n]$, this method shows that in order to find the i^{th} column of A^{-1} (for all $1 \leq i \leq n$) one need only solve the matrix equation $Ax = e_i$

Characterizations of Invertible Matrices

Theorem: Let A be an $n \times n$ matrix. The following are equivalent:

a) A is invertible

b) $A \sim I_n$

c) A has n pivot positions

d) $Ax = 0$ has only the trivial solution

e) The columns of A are linearly independent

f) The linear transformation $T(x) = Ax$ is one-to-one

g) $Ax = b$ has at least one solution for all b in $\mathbb{R}^n$

h) The columns of A span $\mathbb{R}^n$

i) The linear transformation $T(x) = Ax$ is onto

j) There exists $n \times n$ matrix C such that $CA = I_n$

k) There exists $n \times n$ matrix D such that $AD = I_n$

l) A^T is invertible